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ON DUCK SOLUTIONS IN R^4

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ABSTRACT. In this paper, we will prove the existence of duck solutions with winding in the coupled Fitzhugh-Nagumo equation. As the system is described by the slow-fast one in R^4 , we will find the ducks in R^4 .

Let consider the following slow-fast system:

$$(1) \quad \begin{aligned} \epsilon dx_1/dt &= h_1(x_1, x_2, y_1, y_2, u), \\ \epsilon dx_2/dt &= h_2(x_1, x_2, y_1, y_2, u), \\ dy_1/dt &= f(x_1, x_2, y_1, y_2, u), \\ dy_2/dt &= g(x_1, x_2, y_1, y_2, u), \end{aligned}$$

where ϵ is infinitesimally small and u is a parameter. We assume that $H = (h_1, h_2)$ has $\text{rank}H = 2$ at almost every where. In this paper, we put

$$(2) \quad \begin{aligned} h_1 &= y_1 + x_1 - x_1^3/3 + \gamma(x_1 - x_2), \\ h_2 &= y_2 + x_2 - x_2^3/3 + \gamma(x_2 - x_1), \\ f &= -(x_1 - a + by_1)/c, \\ g &= -(x_2 - a + by_2)/c, \end{aligned}$$

and for the simplicity, we put $a = 0$, $\gamma = -1$. So, the parameters are b and c but only b is essential. This slow-fast system is reduced from the coupled Fitzhugh-Nagumo equation proposed by S.A.Campbell[1], 2000. When $\epsilon = 0$, the fast system gives a 2-dim differentiable manifold as a constrained surface. Because of satisfying $\text{rank}H = 2$ regarding especially x_1, x_2 , the system(1) can be reduced to the slow-fast system projected in R^3 :

$$(3) \quad \begin{aligned} dy_1/dt &= -(x_1 + by_1)/c, \\ dy_2/dt &= -(x_1^3/3 - y_1 + by_2)/c, \\ \epsilon dx_1/dt &= y_2 - (x_1^3/3 - y_1)^3/3 + x_1, \end{aligned}$$

under the condition, which $|dx_1/dt - dx_2/dt|$ is limited. On the constrained surface in the system(3), we can get the time scaled reduced system:

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$$\begin{aligned}
 (4) \quad & dy_1/dt = -(x_1 + by_1)(1 - (x_1^3/3 - y_1)^2 x_1^2), \\
 & dy_2/dt = -(x_1^3/3 - y_1 + by_2)(1 - (x_1^3/3 - y_1)^2 x_1^2), \\
 & dx_1/dt = (x_1^3/3 - y_1)^2 (x_1 + by_1) + x_1^3 - y_1 + by_2.
 \end{aligned}$$

Then, the pseudo singular point, that is, the singular point of the system(4) is determined by

$$\begin{aligned}
 (5) \quad & (x_1^3/3 - y_1)^2 (x_1 + by_1) + x_1^3/3 - y_1 + by_2 = 0, \\
 & 1 - (x_1^3/3 - y_1)^2 x_1^2 = 0.
 \end{aligned}$$

Note that the second equation in (5) can be expressed as $x_1^3/3 - y_1 = +(-)1/x_1$. In the case $(-)$, there are 2 pseudo singular points:

$(1, 4/3, -1, -4/3)$, and $(-1, -4/3, 1, 4/3)$.

These points do not depend on the parameter b , therefore they are structurally stable.

As the characteristic equation of the linearized system is

$$(6) \quad \lambda(\lambda - (2 + 8b/3))(\lambda + 8b/3) = 0,$$

we can conclude that these will be node if $-3/4 < b < 0$. Then there are duck solutions at the pseudo-singular node. This fact implies they are winding. See[3], [4], [6].

In the case $(+)$, there are 4 pseudo singular points which depend on the parameter b . The characteristic equation in this case is

$$(7) \quad \lambda(A\lambda^2 + B\lambda + C)/(3 + D)^3 = 0,$$

where

$$A = -D^3 - 27D + 36b^2 - 108$$

$$B = 2[(4b^2 - 9)D^3 + (16b^4 - 90b^2 + 243))D - 162b^2 + 486]/3b$$

$$C = -4[405D^3 + (64b^6 - 720b^4 + 291b^2 - 3645)D + 576b^6 - 3024b^4 + 3888b^2]/9b^2$$

$$D = \sqrt{(3 - 2b)(3 + 2b)}.$$

If $0 < b < 3/2$, there exist the pseudo singular points, as

$$(8) \quad x_1 = +(-)\sqrt{3/2b}, \text{ and } +(-)\sqrt{(9/b^2 - 4)/2}.$$

The eigen values of all four singular points are the same due to the symmetry. They arise in some sort of pitchfork bifurcation from the singular points in the $(-)$ equation at $b = 3/2$. If $0.388 < b < 1.4489$, there will be the ducks at the pseudo-singular node and spirals if $0 < b < 0.388$ or $1.4489 < b < 3/2$.

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